

Supersymmetric Matrix Models and the Meander Problem

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Abstract

We consider matrix-model representations of the meander problem which describes, in particular, combinatorics for foldings of closed polymer chains. We introduce a supersymmetric matrix model for describing the principal meander numbers. This model is of the type proposed by Marinari and Parisi for discretizing a superstring in $D = 1$ while the supersymmetry is realized in $D = 0$ as a rotational symmetry between bosonic and fermionic matrices. Using non-commutative sources, we reformulate the meander problem in a Boltzmannian Fock space whose annihilation and creation operators obey the Cuntz algebra. We discuss also the relation between the matrix models describing the meander problem and the Kazakov–Migdal model on a D -dimensional lattice.

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1 Introduction

Matrix models is a standard tool for describing discretized random surfaces (or, equivalently, strings) [1]. A supersymmetric extension of this construction was first proposed by Marinari and Parisi [2] and studied for the $D = 1$ dimensional target space.

We introduce in the present paper supersymmetric matrix models in the $D = 0$ dimensional target space which differ from the Hermitean supermatrix models of Ref. [3] and are a version of the Marinari–Parisi construction in $D = 0$. Our model deals with the “superfields”

$$W_a = (B, F), \quad \bar{W}_a = (B^\dagger, \bar{F}) \quad (1.1)$$

where $a = 1, 2$ while B and F are complex bosonic and fermionic (*i.e.* Grassmann valued) $N \times N$ matrices, respectively.

Since the propagators for both bosonic and fermionic matrices coincide:

$$\begin{aligned} \langle B_{ij} B_{kl}^\dagger \rangle_{\text{Gauss}} &= \frac{1}{N} \delta_{il} \delta_{kj}, \\ \langle F_{ij} \bar{F}_{kl} \rangle_{\text{Gauss}} &= \frac{1}{N} \delta_{il} \delta_{kj}, \end{aligned} \quad (1.2)$$

the supersymmetry reduces in $D = 0$ simply to rotations between the B - and F -components. The proper transformation reads

$$\begin{aligned} \delta B &= \bar{\epsilon} F, & \delta F &= -\epsilon B, \\ \delta B^\dagger &= \bar{F} \epsilon, & \delta \bar{F} &= -B^\dagger \bar{\epsilon}, \end{aligned} \quad (1.3)$$

where ϵ and $\bar{\epsilon}$ are Grassmann valued.

Any potential, which is symmetrically constructed from the “superfields” (1.1), is supersymmetric so that contributions from the loops of the bosonic and fermionic matrix fields are mutually cancelled which is the key property of the supersymmetry. The simplest Gaussian supersymmetric potential reads

$$V_{\text{Gauss}} = N \sum_{a=1}^2 \text{tr} \bar{W}_a W_a \equiv N \text{tr} (B^\dagger B + \bar{F} F), \quad (1.4)$$

which reproduces the propagators (1.2). It is obviously invariant under the rotation (1.3) even when ϵ and $\bar{\epsilon}$ are fermionic $N \times N$ matrices. It is also clear from Eq. (1.4) why one needs complex matrices in $D = 0$: the trace of the square of a fermionic matrix vanishes.

We elaborate in this paper the technique for dealing with the supersymmetric matrix models on an example of the one which describes combinatorics of the meander numbers. This challenging combinatorial problem, which is described in Sect. 2, is not yet solved. The associated matrix model, which describes a physical problem of enumerating different ways of foldings of a closed polymer chain, is of a next level of complexity with respect to the Hermitean one- or two-matrix models, the multi-matrix chain and its multi-dimensional extension — the Kazakov–Migdal model [4].

We introduce in Sect. 2 the complex matrix model which is equivalent to the Hermitean one [5] in describing the meander numbers. We construct then the supersymmetric matrix model which describes the principal meanders. We discuss also the relation between the matrix models describing the meander problem and the Kazakov–Migdal model on a D -dimensional lattice.

Using non-commutative sources, we reformulate in Sect. 3 the meander problem as a problem of averaging in a Boltzmannian Fock space whose annihilation and creation operators obey the Cuntz algebra. The averaging expression is represented in the form of a product of two continued fractions.

The Appendix A contains a solution of the combinatorial problem of summing over words built up of unitary matrices, which is equivalent to the Kazakov–Migdal model with the Gaussian potential, via free random variables.

In the Appendix B we demonstrate how the equations of Sect. 3, which are obtained using the matrix-model representations, can be alternatively derived pure combinatorially.

We comment in the Appendix C on a possibility of solving the meander problem via free random variables. We show that this approach does not work for the meander problem since the variables are not free for this case so that the theorem of addition of free random variables is not applicable.

2 Matrix models for the meander problem

The meander problem is known to people working on Quantum Field Theory since the middle of the eighties from V. Arnold. The problem is to calculate combinatorial numbers associated with the crossings of an infinite river (Meander) and a closed road by $2n$ bridges.¹ Neither the river nor the road intersects with itself. These principle meander numbers, M_n , obviously describe the number of different foldings of a closed strip of $2n$ stamps or of a closed polymer chain.

One can consider also a generalized problem of the multi-component meander numbers $M_n^{(k)}$ which are associated with k closed loops of the road so that $M_n \equiv M_n^{(1)}$. The results of a computer enumeration of the meander numbers are presented in Refs. [7, 6] up to $n = 12$.

2.1 Hermitean matrix model for meanders

Meanders can be described by the following Hermitean matrix model [5]

$$\mathcal{F}_{N \times N}(c) = \frac{2}{N^2} \int \prod_{a=1}^m dW_a e^{-\frac{N}{2} \sum_{a=1}^m \text{tr} W_a^2} \ln \left(\int d\phi e^{-\frac{N}{2} \text{tr} \phi^2 + \frac{cN}{2} \sum_{a=1}^m \text{tr} (\phi W_a \phi W_a)} \right) \quad (2.1)$$

¹See Ref. [6] for an introduction to the subject.

where the integration goes over the $N \times N$ Hermitean matrices W_a ($a = 1, \dots, m$) and ϕ . The logarithm in Eq. (2.1) leaves only one closed loop of the field ϕ . The coupling constant c is associated with the (quartic) interaction between W_a and ϕ .

Expanding the generating function (2.1) in c and identifying the diagrams with the ones for the meanders, one relates the large- N limit of $\mathcal{F}_{N \times N}(c)$ with the following sum over the meander numbers

$$\lim_{N \rightarrow \infty} \mathcal{F}_{N \times N}(c) = \sum_{n=1}^{\infty} \frac{c^{2n}}{2n} \sum_{k=1}^n M_n^{(k)} m^k. \quad (2.2)$$

The $N \rightarrow \infty$ limit is needed to keep only planar diagrams as in the meander problem.

The RHS of Eq. (2.1) can be expressed entirely via the Gaussian averages of W 's. This leads to the following representation of the meander numbers:

$$\sum_{k=1}^n M_n^{(k)} m^k = \sum_{a_1, a_2, \dots, a_{2n-1}, a_{2n}=1}^m \left\langle \frac{1}{N} \text{tr} W_{a_1} W_{a_2} \cdots W_{a_{2n-1}} W_{a_{2n}} \right\rangle_{\text{Gauss}}^2, \quad (2.3)$$

where the average over W 's is calculated with the Gaussian weight — the same as in (2.1). This formula can be proven by calculating the Gaussian integral over ϕ in Eq. (2.1), expanding the result in c and comparing with the RHS of Eq. (2.2). The factorization at large N is also used.

The principle meander numbers M_n are given by Eq. (2.3) as the linear-in- m -terms, *i.e.* as linear terms of the expansion in m . This looks like the replica trick which suppresses higher loops of the field W .

The ordered but cyclic-symmetric sequence of indices $a_1, a_2, \dots, a_{2n-1}, a_{2n}$ is often called a *word* constructed of m letters. The average on the RHS of Eq. (2.3) is the meaning of a word. Thus, the meander problem is equivalent to summing the squares of all the words with the Gaussian meaning.

The Gaussian averages on the RHS of Eq. (2.2) can be represented, making the Wick pairing, via the Kronecker deltas:

$$\left\langle \frac{1}{N} \text{tr} W_{a_1} W_{a_2} \cdots W_{a_{2n-1}} W_{a_{2n}} \right\rangle_{\text{Gauss}} = \delta_{a_1 a_2} \delta_{a_3 a_4} \cdots \delta_{a_{2n-1} a_{2n}} + \text{planar permutations}. \quad (2.4)$$

The “planar permutations” means here that one should sum up over all the permutations of the indices a_i 's which are consistent with the planarity. This is standard for the large- N limit.

To calculate the meander numbers, one should sum up the square of the RHS of Eq. (2.4) over a_i 's as is prescribed by Eq. (2.3). This is a convenient practical way of calculating the meander numbers.

Since for $m = 1$

$$\left\langle \frac{1}{N} \text{tr} W^{2n} \right\rangle_{\text{Gauss}} = \frac{(2n)!}{(n+1)!n!} \equiv C_n, \quad (2.5)$$

which is known as the Catalan number of the order n , one gets from Eq. (2.3)

$$\sum_{k=1}^n M_n^{(k)} = C_n^2. \quad (2.6)$$

This is nothing but the first sum rule of Ref. [6].

2.2 Complex matrix model for meanders

The combination of deltas on the RHS of Eq. (2.4) can be alternatively represented as the Gaussian average over the complex matrices:

$$\begin{aligned} & \left\langle \frac{1}{N} \text{tr} W_{a_1} W_{a_2}^\dagger \cdots W_{a_{2n-1}} W_{a_{2n}}^\dagger \right\rangle_{\text{Gauss}} \\ &= \int \prod_{a=1}^m dW_a^\dagger dW_a e^{-N \sum_{a=1}^m \text{tr} W_a^\dagger W_a} \frac{1}{N} \text{tr} W_{a_1} W_{a_2}^\dagger \cdots W_{a_{2n-1}} W_{a_{2n}}^\dagger. \end{aligned} \quad (2.7)$$

The generating function, associated with the representation of the meanders via the complex matrices, reads

$$\begin{aligned} \mathcal{F}(c) &= \frac{1}{N^2} \left\langle \ln \left(\int d\phi_1 d\phi_2 e^{-S} \right) \right\rangle_{\text{Gauss}} \\ &\equiv \frac{1}{N^2} \int \prod_{a=1}^m dW_a^\dagger dW_a e^{-N \sum_{a=1}^m \text{tr} W_a^\dagger W_a} \ln \left(\int d\phi_1 d\phi_2 e^{-S} \right) \end{aligned} \quad (2.8)$$

with

$$S = \frac{N}{2} \text{tr} \phi_1^2 + \frac{N}{2} \text{tr} \phi_2^2 - cN \sum_{a=1}^m \text{tr} (\phi_1 W_a^\dagger \phi_2 W_a). \quad (2.9)$$

Here ϕ_1 and ϕ_2 are Hermitean while W_a ($a = 1, \dots, m$) are general complex matrices.

Quite similarly to Eq. (2.1) where the Hermitean matrix ϕ can be represented in a diagonal form $\phi = \text{diag}(\Lambda^{(1)}, \dots, \Lambda^{(N)})$, the matrices ϕ_1 and ϕ_2 in Eqs. (2.8), (2.9) can always be made diagonal:

$$\begin{aligned} \phi_1 &= \Lambda_1 \equiv \text{diag}(\Lambda_1^{(1)}, \dots, \Lambda_1^{(N)}), \\ \phi_2 &= \Lambda_2 \equiv \text{diag}(\Lambda_2^{(1)}, \dots, \Lambda_2^{(N)}). \end{aligned} \quad (2.10)$$

This can be shown representing ϕ_1 and ϕ_2 as

$$\phi_1 = \Omega_1^\dagger \Lambda_1 \Omega_1, \quad \phi_2 = \Omega_2^\dagger \Lambda_2 \Omega_2 \quad (2.11)$$

and absorbing the unitary matrices Ω_1 and Ω_2 by the transformation of W_a :

$$W_a \longrightarrow \Omega_2^\dagger W_a \Omega_1, \quad W_a^\dagger \longrightarrow \Omega_1^\dagger W_a^\dagger \Omega_2. \quad (2.12)$$

The measure $dW_a^\dagger dW_a$ does not change under the transformation (2.12) since W_a are general complex matrices.

It is convenient to introduce one more generating function

$$M(c) = c \left\langle \frac{\int d\phi_1 d\phi_2 e^{-S} \frac{1}{N} \text{tr} \phi_1 W_1^\dagger \phi_2 W_1}{\int d\phi_1 d\phi_2 e^{-S}} \right\rangle_{\text{Gauss}} \quad (2.13)$$

where only one component of W_a , say the first one, enters the averaging expression. Differentiating the generating function (2.8) with respect to c and noting that all m components of W_a are on equal footing, we get the relation

$$c \frac{d\mathcal{F}(c)}{dc} = m M(c) \quad (2.14)$$

between the two generating functions.

In order to show how the complex matrix model recovers the meander numbers, let us replace ϕ_1^{ij} or ϕ_2^{ij} in the numerator of Eq. (2.13) by $N^{-1} \partial / \partial \phi_1^{ji}$ or $N^{-1} \partial / \partial \phi_2^{ji}$, respectively, and integrate by parts. Repeating this procedure iteratively, we get

$$M(c) = \sum_{n=1}^{\infty} c^{2n} \sum_{k=1}^n M_n^{(k)} m^{k-1}, \quad (2.15)$$

with

$$\sum_{k=1}^n M_n^{(k)} m^{k-1} = \sum_{a_2, \dots, a_{2n-1}, a_{2n}=1}^m \left\langle \frac{1}{N} \text{tr} W_1 W_{a_2}^\dagger \cdots W_{a_{2n-1}} W_{a_{2n}}^\dagger \right\rangle_{\text{Gauss}}^2, \quad (2.16)$$

where the Gaussian average is defined by Eq. (2.7). Equation (2.16) can be alternatively derived calculating the Gaussian integrals over ϕ_1 and ϕ_2 in Eq. (2.13) by virtue of

$$\begin{aligned} \int d\phi_1 d\phi_2 e^{-S} &= \int d\phi_2 e^{-\frac{N}{2} \text{tr} \phi_2^2 + \frac{1}{2} c^2 N \sum_{a,b=1}^m \text{tr} (\phi_2 W_a W_b^\dagger \phi_2 W_b W_a^\dagger)} \\ &= \det^{-1/2} \left[\mathbf{I} \otimes \mathbf{I} - c^2 \sum_{a,b=1}^m W_a W_b^\dagger \otimes (W_b W_a^\dagger)^t \right]. \end{aligned} \quad (2.17)$$

For $m = 1$ the formula

$$\left\langle \frac{1}{N} \text{tr} (W W^\dagger)^n \right\rangle_{\text{Gauss}} = C_n, \quad (2.18)$$

which is analogous to Eq. (2.5), holds for the complex matrices. This results again in Eq. (2.6).

It is instructive to consider also the case when W is a fermionic Grassmann valued matrix *à la* Ref. [8]². We shall denote the fermionic matrix as F and its conjugate as \bar{F} . Then we get [8]

$$\left\langle \frac{1}{N} \text{tr} (F \bar{F})^n \right\rangle_{\text{Gauss}} = \begin{cases} 0 & n = 2p \text{ (even)} \\ C_p & n = 2p + 1 \text{ (odd)} \end{cases}. \quad (2.19)$$

²See Ref. [9] for a review

Since each loop of the fermionic field is accompanied by a factor of (-1) , we arrive at the sum rule

$$\sum_{k=1}^n (-)^{k-1} M_n^{(k)} = \begin{cases} 0 & n = 2p \text{ (even)} \\ C_p^2 & n = 2p + 1 \text{ (odd)} \end{cases} . \quad (2.20)$$

This is nothing but the second sum rule of Ref. [6].

Note that the trace of the square of a fermionic matrix vanishes because of the anticommutation relation imposed on the components. This is why we did not consider Hermitean fermionic matrices and used first a representation of meanders in terms of complex matrices to discuss fermionic representation of meanders. Fermionic matrix models are a natural representation of the notion of the signature of arch configurations of Ref. [6].

2.3 General matrix model for meanders

The consideration of the previous Subsection suggests the following representation of the meander numbers via a general complex matrix model which includes both bosonic and fermionic matrices.

Let us consider a general W_a which involves both bosonic (complex) and fermionic (Grassmann) components:

$$\begin{aligned} W_a &= (B_1, B_2, \dots, B_{m_b}, F_1, F_2, \dots, F_{m_f}) , \\ \bar{W}_a &= (B_1^\dagger, B_2^\dagger, \dots, B_{m_b}^\dagger, \bar{F}_1, \bar{F}_2, \dots, \bar{F}_{m_f}) . \end{aligned} \quad (2.21)$$

Here m_b and m_f ($m = m_b + m_f$) are the numbers of the bosonic and fermionic components, respectively. Let us define the generating function $\mathcal{F}(c)$ by the formulas

$$\begin{aligned} \mathcal{F}(c) &= \frac{1}{N^2} \left\langle \ln \left(\int d\phi_1 d\phi_2 e^{-S} \right) \right\rangle_{\text{Gauss}} \\ &\equiv \frac{1}{N^2} \int \prod_{a=1}^m d\bar{W}_a dW_a e^{-N \sum_{a=1}^m \text{tr} \bar{W}_a W_a} \ln \left(\int d\phi_1 d\phi_2 e^{-S} \right) \end{aligned} \quad (2.22)$$

and

$$S = \frac{N}{2} \text{tr} \phi_1^2 + \frac{N}{2} \text{tr} \phi_2^2 - cN \sum_{a=1}^m \text{tr} (\phi_1 \bar{W}_a \phi_2 W_a) , \quad (2.23)$$

which generalize Eqs. (2.8) and (2.9). Then, the generating function (2.22) is related to the meander numbers by

$$\lim_{N \rightarrow \infty} \mathcal{F}_{N \times N}(c) = \sum_{n=1}^{\infty} \frac{c^{2n}}{2n} \sum_{k=1}^n M_n^{(k)} (m_b - m_f)^k . \quad (2.24)$$

Here m_f emerges with the minus sign since fermion loops are always accompanied with the minus sign.

Equation (2.16) is extended to the given general case of both bosonic and fermionic matrices as

$$\sum_{k=1}^n M_n^{(k)}(m_b - m_f)^{k-1} = \sum_{a_2, \dots, a_{2n-1}, a_{2n}=1}^m \left\langle \frac{1}{N} \text{tr} W_1 \bar{W}_{a_2} \cdots W_{a_{2n-1}} \bar{W}_{a_{2n}} \right\rangle_{\text{Gauss}} \left\langle \frac{1}{N} \text{tr} \bar{W}_{a_{2n}} W_{a_{2n-1}} \cdots \bar{W}_{a_2} W_1 \right\rangle_{\text{Gauss}}. \quad (2.25)$$

Analogously, Eqs. (2.13), (2.14) and (2.15) are generalized as

$$M(c) = c \left\langle \frac{\int d\phi_1 d\phi_2 e^{-S} \frac{1}{N} \text{tr} \phi_1 \bar{W}_1 \phi_2 W_1}{\int d\phi_1 d\phi_2 e^{-S}} \right\rangle_{\text{Gauss}}, \quad (2.26)$$

$$c \frac{d\mathcal{F}(c)}{dc} = (m_b - m_f) M(c), \quad (2.27)$$

where

$$M(c) = \sum_{n=1}^{\infty} c^{2n} \sum_{k=1}^n M_n^{(k)}(m_b - m_f)^{k-1}. \quad (2.28)$$

Equations (2.13), (2.14) and (2.15) are obviously reproduced when $m_f = 0$.

In order to prove Eqs. (2.25), (2.28), let us first note that Eq. (2.17) is extended to the general case of both bosonic and fermionic components as

$$\begin{aligned} \int d\phi_1 d\phi_2 e^{-S} &= \int d\phi_2 e^{-\frac{N}{2} \text{tr} \phi_2^2 + \frac{1}{2} c^2 N \sum_{a,b=1}^m \text{tr} (\bar{W}_a \phi_2 W_a \bar{W}_b \phi_2 W_b)} \\ &= \int d\phi_2 e^{-\frac{N}{2} \text{tr} \phi_2^2 + \frac{1}{2} c^2 N \sum_{a,b=1}^m \sigma(a) \text{tr} (\phi_2 W_a \bar{W}_b \phi_2 W_b \bar{W}_a)} \\ &= \det^{-1/2} \left[\mathbf{I} \otimes \mathbf{I} - c^2 \sum_{a,b=1}^m \sigma(a) W_a \bar{W}_b \otimes (W_b \bar{W}_a)^{\dagger} \right], \end{aligned} \quad (2.29)$$

where

$$\sigma(a) = \begin{cases} 1 & \text{for } B \\ -1 & \text{for } F \end{cases} \quad (2.30)$$

is the signature factor of the component W_a .

Expanding the determinant (2.29) on the RHS of Eq. (2.22) in c^2 , we get the following representation

$$\begin{aligned} \lim_{N \rightarrow \infty} \mathcal{F}_{N \times N}(c) &= \sum_{n=1}^{\infty} \frac{c^{2n}}{2n} \sum_{a_1, \dots, a_{2n-1}, a_{2n}=1}^m \sigma(a_1) \sigma(a_3) \cdots \sigma(a_{2n-1}) \\ &\times \left\langle \frac{1}{N} \text{tr} W_{a_1} \bar{W}_{a_2} \cdots W_{a_{2n-1}} \bar{W}_{a_{2n}} \right\rangle_{\text{Gauss}} \left\langle \frac{1}{N} \text{tr} W_{a_{2n}} \bar{W}_{a_{2n-1}} \cdots W_{a_2} \bar{W}_{a_1} \right\rangle_{\text{Gauss}} \\ &= \sum_{n=1}^{\infty} \frac{c^{2n}}{2n} \sum_{a_1, \dots, a_{2n-1}, a_{2n}=1}^m \\ &\times \left\langle \frac{1}{N} \text{tr} W_{a_1} \bar{W}_{a_2} \cdots W_{a_{2n-1}} \bar{W}_{a_{2n}} \right\rangle_{\text{Gauss}} \left\langle \frac{1}{N} \text{tr} \bar{W}_{a_{2n}} W_{a_{2n-1}} \cdots \bar{W}_{a_2} W_{a_1} \right\rangle_{\text{Gauss}} \end{aligned} \quad (2.31)$$

where the signs of the fermionic components are transformed at the last step using

$$\begin{aligned} & \sigma(a_1)\sigma(a_3)\cdots\sigma(a_{2n-1}) \left\langle \frac{1}{N} \text{tr } W_{a_1} \bar{W}_{a_2} \cdots W_{a_{2n-1}} \bar{W}_{a_{2n}} \right\rangle_{\text{Gauss}} \\ &= \left\langle \frac{1}{N} \text{tr } \bar{W}_{a_1} W_{a_3} \cdots \bar{W}_{a_{2n-1}} W_{a_{2n}} \right\rangle_{\text{Gauss}} . \end{aligned} \quad (2.32)$$

Therefore, the order of matrices in Eq. (2.25) is chosen in the way to absorb the signature factors for the fermionic components.

2.4 Supersymmetric matrix model for principle meander

Having the representation (2.25) of meanders via general complex matrices (either bosonic or fermionic), we can utilize the idea of supersymmetry to kill the loops of the W -field instead of the replica trick. Let us consider the two-component W_a whose first component is bosonic while the second one is a fermionic matrix:

$$W_a = (B, F) , \quad \bar{W}_a \equiv W_a^\dagger = (B^\dagger, \bar{F}) , \quad (2.33)$$

that is given by Eq. (2.21) with $m_b = m_f = 1$.

The generating function (2.22) equals zero for the supersymmetric model since all the loops of the B and F fields are mutually cancelled. One should use alternatively the generating function (2.26) which can be represented for the supersymmetric matrix model as

$$M(c) = \left\langle \frac{1}{N} \text{tr } B B^\dagger \ln \left(\int d\phi_1 d\phi_2 e^{-S} \right) \right\rangle_{\text{Gauss}} . \quad (2.34)$$

Here S is explicitly given by

$$S = \frac{N}{2} \text{tr } \phi_1^2 + \frac{N}{2} \text{tr } \phi_2^2 - cN \text{tr } (\phi_1 B^\dagger \phi_2 B) - cN \text{tr } (\phi_1 \bar{F} \phi_2 F) \quad (2.35)$$

as is prescribed by Eq. (2.23) with W_a substituted according to Eq. (2.33). The equivalence of Eqs. (2.26) and (2.34) in the supersymmetric case can be proven replacing B in the integrand on the RHS of Eq. (2.34) by $N^{-1} \partial / \partial B^\dagger$, integrating by parts, and recalling that $\mathcal{F}(c) = 0$.

All the multi-component meanders in Eqs. (2.25), (2.28) vanish in the supersymmetric case and we get the following representation for the principle meander

$$M(c) = \sum_{n=1}^{\infty} c^{2n} M_n \quad (2.36)$$

with

$$\begin{aligned} M_n &= \sum_{a_2, \dots, a_{2n-1}, a_{2n}=1}^2 \\ &\times \left\langle \frac{1}{N} \text{tr } B \bar{W}_{a_2} \cdots W_{a_{2n-1}} \bar{W}_{a_{2n}} \right\rangle_{\text{Gauss}} \left\langle \frac{1}{N} \text{tr } \bar{W}_{a_{2n}} W_{a_{2n-1}} \cdots \bar{W}_{a_2} B \right\rangle_{\text{Gauss}} , \end{aligned} \quad (2.37)$$

n	Structure	Combinatorics	Value	Contribution to M_n
1	$\langle B^2 \rangle$	1	1	1
2	$\langle B^4 \rangle$ $\langle B^2 F^2 \rangle$	$\left. \begin{matrix} 1 \\ 2 \end{matrix} \right\} 3$	$\begin{matrix} 2 \\ 1 \end{matrix}$	$\left. \begin{matrix} 4 \\ -2 \end{matrix} \right\} 2$
3	$\langle B^6 \rangle$ $\langle B^4 F^2 \rangle$ $\langle B^2 F B^2 F \rangle$ $\langle B^2 F^4 \rangle$ $\langle B F^2 B F^2 \rangle$	$\left. \begin{matrix} 1 \\ 4 \\ 2 \\ 2 \\ 1 \end{matrix} \right\} 10$	$\begin{matrix} 5 \\ 2 \\ 1 \\ 0 \\ 1 \end{matrix}$	$\left. \begin{matrix} 25 \\ -16 \\ -2 \\ 0 \\ 1 \end{matrix} \right\} 8$
4	$\langle B^8 \rangle$ $\langle B^6 F^2 \rangle$ $\langle B^4 F B^2 F \rangle$ $\langle B^4 F^4 \rangle$ $\langle B^2 F^2 B^2 F^2 \rangle$ $\langle B^3 F^2 B F^2 \rangle$ $\langle B^2 F B^2 F^3 \rangle$ $\langle B^2 F^6 \rangle$ $\langle B F^2 B F^4 \rangle$ $\langle B F^2 B F B^2 F \rangle$	$\left. \begin{matrix} 1 \\ 6 \\ 6 \\ 4 \\ 2 \\ 4 \\ 4 \\ 2 \\ 2 \\ 4 \end{matrix} \right\} 35$	$\begin{matrix} 14 \\ 5 \\ 2 \\ 0 \\ 1 \\ 2 \\ 0 \\ 1 \\ 0 \\ 1 \end{matrix}$	$\left. \begin{matrix} 196 \\ -150 \\ -24 \\ 0 \\ 2 \\ 16 \\ 0 \\ -2 \\ 0 \\ 4 \end{matrix} \right\} 42$

Table 1: Calculation of the principle meander numbers in the supersymmetric matrix model (2.37) up to $n = 4$.

where we kept trace of the order of matrices how it appears from Eq. (2.25). The signs, which are essential for the fermionic components, have been transformed using the formula (2.32) with $\sigma(a)$ being the signature factor of the component W_a defined by Eq. (2.30).

Equation (2.37) is a nice representation of the principle meander numbers which looks more natural than the one based on the replica trick. A hope is that it will be simpler to solve the $m = 2$ supersymmetric model than a pure bosonic one at arbitrary m . How the representation (2.37) reproduces the principle meander numbers is illustrated by the Table 1 up to $n = 4$.

Alternatively, one can calculate

$$\begin{aligned}
-M_n = & \sum_{a_2, \dots, a_{2n-1}, a_{2n}=1}^2 \\
& \times \left\langle \frac{1}{N} \text{tr} F \bar{W}_{a_2} \cdots W_{a_{2n-1}} \bar{W}_{a_{2n}} \right\rangle_{\text{Gauss}} \left\langle \frac{1}{N} \text{tr} \bar{W}_{a_{2n}} W_{a_{2n-1}} \cdots \bar{W}_{a_2} F \right\rangle_{\text{Gauss}}. \quad (2.38)
\end{aligned}$$

The results are presented in the Table 2.

The analogous results for the pure bosonic case given by Eq. (2.16) at $m = 2$ are presented in the Table 3.

n	Structure	Combinatorics	Value	Contribution to $-M_n$
1	$\langle F^2 \rangle$	1	1	-1
2	$\langle F^4 \rangle$ $\langle F^2 B^2 \rangle$	$\left. \begin{matrix} 1 \\ 2 \end{matrix} \right\} 3$	$\begin{matrix} 0 \\ 1 \end{matrix}$	$\left. \begin{matrix} 0 \\ -2 \end{matrix} \right\} -2$
3	$\langle F^6 \rangle$ $\langle F^4 B^2 \rangle$ $\langle F^2 B F^2 B \rangle$ $\langle F^2 B^4 \rangle$ $\langle F B^2 F B^2 \rangle$	$\left. \begin{matrix} 1 \\ 4 \\ 2 \\ 2 \\ 1 \end{matrix} \right\} 10$	$\begin{matrix} 1 \\ 0 \\ 1 \\ 2 \\ 1 \end{matrix}$	$\left. \begin{matrix} -1 \\ 0 \\ 2 \\ -8 \\ -1 \end{matrix} \right\} -8$
4	$\langle F^8 \rangle$ $\langle F^6 B^2 \rangle$ $\langle F^4 B F^2 B \rangle$ $\langle F^4 B^4 \rangle$ $\langle F^2 B^2 F^2 B^2 \rangle$ $\langle F^3 B^2 F B^2 \rangle$ $\langle F^2 B F^2 B^3 \rangle$ $\langle F^2 B^6 \rangle$ $\langle F B^2 F B^4 \rangle$ $\langle F B^2 F B F^2 B \rangle$	$\left. \begin{matrix} 1 \\ 6 \\ 6 \\ 4 \\ 2 \\ 4 \\ 4 \\ 2 \\ 2 \\ 4 \end{matrix} \right\} 35$	$\begin{matrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 2 \\ 5 \\ 2 \\ 1 \end{matrix}$	$\left. \begin{matrix} 0 \\ -6 \\ 0 \\ 0 \\ 2 \\ 0 \\ 16 \\ -50 \\ -8 \\ 4 \end{matrix} \right\} -42$

Table 2: Same as in the Table 1 but using Eq. (2.38).

The total number of nonvanishing terms on the RHS of Eq. (2.16) for the pure bosonic case, which we shall denote as $\#_n$, is given by the following generating function

$$\sum_{n=0}^{\infty} \#_n c^{2n} = \frac{\frac{m}{2} \sqrt{1 - 4(m-1)c^2} - \frac{m}{2} + 1}{1 - c^2 m^2}. \quad (2.39)$$

This formula is derived in the Appendix A using non-commutative free random variables.

For $m = 2$ Eq. (2.39) yields

$$\sum_{n=0}^{\infty} \#_n c^{2n} = \frac{1}{\sqrt{1 - 4c^2}} = \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} c^{2n}. \quad (2.40)$$

These numbers describe the sum of the combinatorial numbers in the third column of the Tables 1 – 3.

2.5 Relation to the Kazakov–Migdal model

Equation (2.39) is known from the solution of the Kazakov–Migdal model with the Gaussian potential [10]. There is the following reason for that. Suppose that the matrices W_a are unitary instead of the general complex ones. Then one has

$$\left\langle \frac{1}{N} \text{tr} U_{a_1} U_{a_2}^\dagger \cdots U_{a_{2n-1}} U_{a_{2n}}^\dagger \right\rangle_{\text{Haar measure}} = \begin{cases} 1 & \text{for closed loops} \\ 0 & \text{for open loops} \end{cases}. \quad (2.41)$$

n	Structure	Combinatorics	Value	Contribution
1	$\langle A^2 \rangle$	1	1	1
2	$\langle A^4 \rangle$ $\langle A^2 B^2 \rangle$	$\left. \begin{matrix} 1 \\ 2 \end{matrix} \right\} 3$	$\begin{matrix} 2 \\ 1 \end{matrix}$	$\left. \begin{matrix} 4 \\ 2 \end{matrix} \right\} 6$
3	$\langle A^6 \rangle$ $\langle A^4 B^2 \rangle$ $\langle A^2 B A^2 B \rangle$ $\langle A^2 B^4 \rangle$ $\langle A B^2 A B^2 \rangle$	$\left. \begin{matrix} 1 \\ 4 \\ 2 \\ 2 \\ 1 \end{matrix} \right\} 10$	$\begin{matrix} 5 \\ 2 \\ 1 \\ 2 \\ 1 \end{matrix}$	$\left. \begin{matrix} 25 \\ 16 \\ 2 \\ 8 \\ 1 \end{matrix} \right\} 52$
4	$\langle A^8 \rangle$ $\langle A^6 B^2 \rangle$ $\langle A^4 B A^2 B \rangle$ $\langle A^4 B^4 \rangle$ $\langle A^2 B^2 A^2 B^2 \rangle$ $\langle A^3 B^2 A B^2 \rangle$ $\langle A^2 B A^2 B^3 \rangle$ $\langle A^2 B^6 \rangle$ $\langle A B^2 A B^4 \rangle$ $\langle A B^2 A B A^2 B \rangle$	$\left. \begin{matrix} 1 \\ 6 \\ 6 \\ 4 \\ 2 \\ 4 \\ 4 \\ 2 \\ 2 \\ 2 \\ 4 \end{matrix} \right\} 35$	$\begin{matrix} 14 \\ 5 \\ 2 \\ 4 \\ 3 \\ 2 \\ 2 \\ 5 \\ 2 \\ 2 \\ 1 \end{matrix}$	$\left. \begin{matrix} 196 \\ 150 \\ 24 \\ 64 \\ 18 \\ 16 \\ 16 \\ 50 \\ 8 \\ 8 \\ 4 \end{matrix} \right\} 546$

Table 3: Same as in the Tables 1 and 2 but using Eq. (2.16) at $m = 2$. The components of W_a are denoted as $W_1 = A, W_2 = B$.

Here the loops represent the sequences of indices $\{a_1, a_2, \dots, a_{2n-1}, a_{2n}\}$. The nonvanishing result is only when the loop is closed and encloses a surface of the vanishing minimal area, *i.e.* each link of the loop is passed at least twice. This is analogous to the so-called local confinement in the Kazakov–Migdal model.

The generating function (2.39) coincides with the following correlator in the Kazakov–Migdal model with the Gaussian potential on an infinite D -dimensional lattice:

$$\begin{aligned}
\sum_{n=0}^{\infty} \#_n c^{2n} &= \left\langle \frac{1}{N} \text{tr} \phi^2(0) \right\rangle \\
&\equiv \frac{\int \prod_x d\phi(x) \prod_{\mu=1}^D dU_{\mu}(x) e^{-S[\phi, U]} \frac{1}{N} \text{tr} \phi^2(0)}{\int \prod_x d\phi(x) \prod_{\mu=1}^D dU_{\mu}(x) e^{-S[\phi, U]}}
\end{aligned} \tag{2.42}$$

with the action

$$S[\phi, U] = N \sum_x \left(\frac{1}{2} \text{tr} \phi_x^2 - c \sum_{\mu=1}^D \text{tr} \left(\phi(x) U_{\mu}^{\dagger}(x) \phi(x + \hat{\mu}) U_{\mu}(x) \right) \right), \tag{2.43}$$

provided that $2D = m$. The integration over the unitary matrices $U_{\mu}(x)$ goes over the Haar measure.

The solution of the Kazakov–Migdal model with the Gaussian potential can be completely reformulated as a combinatorial problem of summing over all closed loops of a

given length with all possible backtrackings (or foldings) included. Its solution [11] is given by Eq. (2.39).

By virtue of the Eguchi–Kawai reduction [12]³, the correlator (2.42) in the Kazakov–Migdal model on the infinite lattice is equivalent to that in the reduced model given by

$$\sum_{n=0}^{\infty} \#_n c^{2n} = \frac{\int d\phi_1 d\phi_2 \prod_{a=1}^m dU_a e^{-S[\phi, U]} \frac{1}{N} \text{tr} \phi_1^2}{\int d\phi_1 d\phi_2 \prod_{a=1}^m dU_a e^{-S[\phi, U]}}, \quad (2.44)$$

with $m = 2D$ and the reduced action being

$$S[\phi, U] = \frac{N}{2} \text{tr} \phi_1^2 + \frac{N}{2} \text{tr} \phi_2^2 - cN \sum_{a=1}^m \text{tr} (\phi_1 U_a^\dagger \phi_2 U_a). \quad (2.45)$$

We have introduced here the index a running from 1 to $m = 2D$ for the reduced model to distinguish from the index μ running from 1 to D for the Kazakov–Migdal model on the infinite lattice. The representation (2.44), (2.45) can be finally rewritten as

$$\sum_{n=1}^{\infty} \#_n c^{2n} = c \left\langle \frac{\int d\phi_1 d\phi_2 e^{-S[\phi, U]} \sum_{a=1}^m \frac{1}{N} \text{tr} \phi_1 U_a^\dagger \phi_2 U_a}{\int d\phi_1 d\phi_2 e^{-S[\phi, U]}} \right\rangle_{\text{Haar measure}}. \quad (2.46)$$

In order to prove the equivalence of Eqs. (2.44) and (2.46), we calculate the integral over $d\phi_1$ and $d\phi_2$ using Eq. (2.17) with W_a substituted by U_a :

$$\int d\phi_1 d\phi_2 e^{-S} = \det^{-1/2} \left[\mathbf{I} \otimes \mathbf{I} - c^2 \sum_{a,b=1}^m U_a U_b^\dagger \otimes (U_b U_a^\dagger)^\dagger \right]. \quad (2.47)$$

The determinants in the numerator and the denominator on the RHS of Eq. (2.46) obviously cancel before the averaging over U_a 's. An analogous cancellation happens in Eq. (2.44) as well in spite of the fact that each of them is averaged over its own U_a 's. The point is that the determinant (2.47) under the sign of averaging over U behaves at large N as a U -independent constant.

The representation (2.46) looks very similar to the generating function (2.13) of the meander numbers. The difference is that the average is over the unitary matrices in Eq. (2.46) and over the Gaussian complex matrices in Eq. (2.13).

We can interpolate between the two cases by modifying the weight for averaging over W 's along the line of Ref. [14]. Let us introduce

$$\langle F[W, W^\dagger] \rangle_\alpha \equiv \int \prod_{a=1}^m \left(dW_a^\dagger dW_a e^{-\frac{\alpha N}{2} \text{tr} \left(W_a^\dagger W_{a-1} + \frac{1}{\alpha} \right)^2 + \frac{N}{2\alpha}} \right) F[W, W^\dagger]. \quad (2.48)$$

Then the averaging over the Gaussian complex matrices is reproduced as $\alpha \rightarrow 0$ while the average over the unitary matrices is recovered as $\alpha \rightarrow \infty$ since the matrix W_α is forced to be unitary as $\alpha \rightarrow \infty$.

³See Ref. [13] for a review.

We see, thus, that the words are the same both for the meander problem and for the Kazakov–Migdal model. The only difference resides in the meaning of nonvanishing words — it equals unity for the unitary matrices.

3 Representation via non-commutative variables

The set $\mathbf{u}_a, \mathbf{u}_a^\dagger$ of non-commutative variables obey the Cuntz algebra

$$\mathbf{u}_a \mathbf{u}_b^\dagger = \delta_{ab} . \quad (3.1)$$

It is convenient to consider them, respectively, as annihilation and creation operators in a Hilbert space with the vacuum $|\Omega\rangle$ which satisfies

$$\begin{aligned} \mathbf{u}_a |\Omega\rangle &= 0 , & \langle\Omega| \mathbf{u}_a^\dagger &= 0 , \\ \langle\Omega|\Omega\rangle &= 1 . \end{aligned} \quad (3.2)$$

The completeness condition says that

$$\sum_{a=1}^m \mathbf{u}_a^\dagger \mathbf{u}_a = 1 - |\Omega\rangle \langle\Omega| . \quad (3.3)$$

There are no more relations between the non-commutative variables.

3.1 Bosonic case

Let us construct the generating function for words via the non-commutative sources \mathbf{u}_a as

$$\begin{aligned} G_\lambda(\mathbf{u}) &= \left\langle \frac{1}{N} \text{tr} \frac{1}{\lambda - \sum_{a=1}^m \mathbf{u}_a W_a} \right\rangle_{\text{Gauss}} \\ &= \sum_{n=0}^{\infty} \frac{1}{\lambda^{2n+1}} \sum_{a_1, a_2, \dots, a_{2n}=1}^m \mathbf{u}_{a_1} \mathbf{u}_{a_2} \cdots \mathbf{u}_{a_{2n}} \left\langle \frac{1}{N} \text{tr} W_{a_1} W_{a_2} \cdots W_{a_{2n}} \right\rangle_{\text{Gauss}} \end{aligned} \quad (3.4)$$

where the average over $a = 1, \dots, m$ matrices W_a is with the Gaussian weight as before.

Then the generating function (2.13) for the meander numbers is given by

$$\langle\Omega| G_\lambda(\mathbf{u}) G_\lambda(\mathbf{u}^\dagger) |\Omega\rangle = c + cmM(c) \quad (3.5)$$

with $c = 1/\lambda^2$. The contraction of indices on the RHS of Eq. (2.3) is obviously reproduced using Eqs. (3.1) and (3.2).

The generating function $G_\lambda(\mathbf{u})$ obeys the Schwinger-Dyson equation

$$\lambda G_\lambda(\mathbf{u}) - 1 = \sum_{a=1}^m G_\lambda(\mathbf{u}) \mathbf{u}_a G_\lambda(\mathbf{u}) \mathbf{u}_a , \quad (3.6)$$

which can be derived in a usual way, by shifting W . The cyclic symmetry of the trace implies

$$\lambda G_\lambda(\mathbf{u}) - 1 = \sum_{b=1}^m \mathbf{u}_b (\lambda G_\lambda(\mathbf{u}) - 1) \mathbf{u}_b^\dagger. \quad (3.7)$$

Inserting here Eq. (3.6), we get

$$\sum_{a=1}^m G_\lambda(\mathbf{u}) \mathbf{u}_a G_\lambda(\mathbf{u}) \mathbf{u}_a = \sum_{a=1}^m \mathbf{u}_a G_\lambda(\mathbf{u}) \mathbf{u}_a G_\lambda(\mathbf{u}). \quad (3.8)$$

An alternative combinatorial derivation of Eqs. (3.5) and (3.6) is presented in the Appendix B.

We shall use for $G_\lambda(\mathbf{u})$ and $G_\lambda(\mathbf{u}^\dagger)$ the short-hand notation

$$G_\lambda \equiv G_\lambda(\mathbf{u}), \quad G_\lambda^\dagger \equiv G_\lambda(\mathbf{u}^\dagger), \quad (3.9)$$

so that Eq. (3.6) can be rewritten as

$$\begin{aligned} \lambda G_\lambda - 1 &= G_\lambda \mathbf{u}_a G_\lambda \mathbf{u}_a, \\ \lambda G_\lambda^\dagger - 1 &= \mathbf{u}_a^\dagger G_\lambda^\dagger \mathbf{u}_a^\dagger G_\lambda^\dagger, \end{aligned} \quad (3.10)$$

where the summation over repeated indices is implied here and below except when it is specially indicated. Using Eq. (3.1), one alternatively rewrites Eq. (3.10) as

$$\begin{aligned} (\lambda G_\lambda - 1) \mathbf{u}_a^\dagger &= G_\lambda \mathbf{u}_a G_\lambda, \\ \mathbf{u}_a (\lambda G_\lambda^\dagger - 1) &= G_\lambda^\dagger \mathbf{u}_a^\dagger G_\lambda^\dagger. \end{aligned} \quad (3.11)$$

If u 's were ordinary commutative variables, the solution to the quadratic equation (3.6) would be simple

$$G_\lambda = \frac{\lambda - \sqrt{\lambda^2 - 4 \vec{u}^2}}{2 \vec{u}^2}. \quad (3.12)$$

For $m = 1$ this is nothing but Wigner's semicircle law and one reproduces Eq. (2.18) by expanding in $1/\lambda$.

A formal solution for the non-commutative variables can be obtained representing Eq. (3.6) as

$$G_\lambda = \frac{1}{\lambda - \mathbf{u}_a G_\lambda \mathbf{u}_a}. \quad (3.13)$$

Iterations of this equation, as was found by Cvitanović [15], lead in the continued fraction

$$G_\lambda(\mathbf{u}) = \frac{1}{\lambda - \mathbf{u}_{a_1} \frac{1}{\lambda - \mathbf{u}_{a_2} \frac{1}{\lambda - \mathbf{u}_{a_3} \frac{1}{\ddots} \mathbf{u}_{a_3}} \mathbf{u}_{a_2}} \mathbf{u}_{a_1}}. \quad (3.14)$$

Expanding the RHS in $1/\lambda^2$, one gets

$$G_\lambda(\mathbf{u}) = \frac{1}{\lambda} + \frac{\mathbf{u}^2}{\lambda^3} + \frac{\mathbf{u}^2\mathbf{u}^2 + \mathbf{u}_a\mathbf{u}^2\mathbf{u}_a}{\lambda^5} + \frac{\mathbf{u}^2\mathbf{u}^2\mathbf{u}^2 + \mathbf{u}_a\mathbf{u}^2\mathbf{u}^2\mathbf{u}_a + \mathbf{u}^2\mathbf{u}_a\mathbf{u}^2\mathbf{u}_a + \mathbf{u}_a\mathbf{u}^2\mathbf{u}_a\mathbf{u}^2 + \mathbf{u}_a\mathbf{u}_b\mathbf{u}^2\mathbf{u}_b\mathbf{u}_a}{\lambda^7} + \mathcal{O}\left(\frac{1}{\lambda^9}\right) \quad (3.15)$$

where \mathbf{u}^2 stands for $\mathbf{u}_a\mathbf{u}_a$. All possible planar combinations of \mathbf{u} 's appear to next orders (with unit coefficients) while the total number of terms to the order λ^{-2n-1} equals the Catalan number C_n . Technically, it is more simple to derive the expansion of $G_\lambda(\mathbf{u})$ in $1/\lambda$ by direct iterations of Eq. (3.6). The substitution into Eq. (3.5) then recovers the lower meander numbers.

Equation (3.6) is in fact well-known for the Gaussian models. It is a consequence of the relation between the generating functionals for all planar graphs $G_\lambda(\mathbf{u})$ and for connected planar graphs

$$\begin{aligned} \mathcal{W}(\mathbf{j}) &= \left\langle \frac{1}{N} \text{tr} \frac{1}{\lambda - \sum_{a=1}^m \mathbf{j}_a W_a} \right\rangle_{\text{conn}} \\ &= \sum_{n=0}^{\infty} \frac{1}{\lambda^{2n+1}} \sum_{a_1, a_2, \dots, a_{2n}=1}^m \mathbf{j}_{a_1} \mathbf{j}_{a_2} \cdots \mathbf{j}_{a_{2n}} \left\langle \frac{1}{N} \text{tr} W_{a_1} W_{a_2} \cdots W_{a_{2n}} \right\rangle_{\text{conn}} \end{aligned} \quad (3.16)$$

which says [16, 15]

$$G(\mathbf{u}) = \mathcal{W}(\lambda \mathbf{u} G(\mathbf{u})) \quad (3.17)$$

while the cyclic symmetry gives

$$\mathcal{W}(\lambda \mathbf{u} G(\mathbf{u})) = \mathcal{W}(\lambda G(\mathbf{u}) \mathbf{u}). \quad (3.18)$$

There is only one connected graph in the Gaussian case so that $\mathcal{W}(\mathbf{j})$ is quadratic

$$\mathcal{W}(\mathbf{j}) = \frac{1}{\lambda} + \frac{\mathbf{j}_a \mathbf{j}_a}{\lambda^3}. \quad (3.19)$$

Equations (3.17) and (3.18) now recover Eqs. (3.6) and (3.8), respectively.

It would be interesting to apply the theory of non-commutative free random variables [17], whose application to matrix models has been discussed recently in Refs. [18, 19], for this problem.

3.2 General case

For a general complex matrix model when some components of W_a are bosonic and some are fermionic, we define the generating functions

$$\begin{aligned} G_\lambda(\mathbf{u}) &= \left\langle \frac{1}{N} \text{tr} \frac{\lambda}{\lambda^2 - \sum_{a,b=1}^m \mathbf{u}_a \mathbf{u}_b W_a \bar{W}_b} \right\rangle_{\text{Gauss}} = \\ &= \sum_{n=0}^{\infty} \frac{1}{\lambda^{2n+1}} \sum_{a_1, a_2, \dots, a_{2n}=1}^m \mathbf{u}_{a_1} \mathbf{u}_{a_2} \cdots \mathbf{u}_{a_{2n-1}} \mathbf{u}_{a_{2n}} \left\langle \frac{1}{N} \text{tr} W_{a_1} \bar{W}_{a_2} \cdots W_{a_{2n-1}} \bar{W}_{a_{2n}} \right\rangle_{\text{Gauss}} \end{aligned} \quad (3.20)$$

and

$$\begin{aligned}\bar{G}_\lambda(\mathbf{u}) &= \left\langle \frac{1}{N} \text{tr} \frac{\lambda}{\lambda^2 - \sum_{a,b=1}^m \mathbf{u}_a \mathbf{u}_b \bar{W}_a W_b} \right\rangle_{\text{Gauss}} = \\ &= \sum_{n=0}^{\infty} \frac{1}{\lambda^{2n+1}} \sum_{a_1, a_2, \dots, a_{2n}=1}^m \mathbf{u}_{a_1} \mathbf{u}_{a_2} \cdots \mathbf{u}_{a_{2n-1}} \mathbf{u}_{a_{2n}} \left\langle \frac{1}{N} \text{tr} \bar{W}_{a_1} W_{a_2} \cdots \bar{W}_{a_{2n-1}} W_{a_{2n}} \right\rangle_{\text{Gauss}}\end{aligned}\quad (3.21)$$

where the non-commutative sources \mathbf{u}_a are the same as before.

The generating functions $G_\lambda(\mathbf{u})$ and $\bar{G}_\lambda(\mathbf{u})$ are not independent due to Eq. (2.32). One gets

$$\bar{G}_\lambda(\mathbf{u}_a) = G_\lambda(\bar{\mathbf{u}}_a) \quad (3.22)$$

where

$$\bar{\mathbf{u}}_a \equiv \sqrt{\sigma(a)} \mathbf{u}_a \quad (3.23)$$

and $\sigma(a)$ is defined by Eq. (2.30). In other words the components of \mathbf{u}_a which are associated with fermionic components of W_a should be multiplied by i on the RHS of Eq. (3.22) while the ones for the bosonic components remain unchanged.

The Schwinger-Dyson equations for $G_\lambda(\mathbf{u})$ and $\bar{G}_\lambda(\mathbf{u})$ can be derived in a usual way from the recurrence relations

$$\begin{aligned}& \left\langle \frac{1}{N} \text{tr} W_{a_1} \bar{W}_{a_2} \cdots W_{a_{2n-1}} \bar{W}_{a_{2n}} \right\rangle_{\text{Gauss}} \\ &= \sum_{k=0}^{n-1} \delta_{a_{2n} a_{2k+1}} \left\langle \frac{1}{N} \text{tr} W_{a_1} \bar{W}_{a_2} \cdots \bar{W}_{a_{2k}} \right\rangle_{\text{Gauss}} \left\langle \frac{1}{N} \text{tr} \bar{W}_{a_{2k+2}} \cdots W_{a_{2n-1}} \right\rangle_{\text{Gauss}}\end{aligned}\quad (3.24)$$

and

$$\begin{aligned}& \left\langle \frac{1}{N} \text{tr} W_{a_1} \bar{W}_{a_2} \cdots W_{a_{2n-1}} \bar{W}_{a_{2n}} \right\rangle_{\text{Gauss}} \\ &= \sum_{k=1}^n \delta_{a_1 a_{2k}} \left\langle \frac{1}{N} \text{tr} \bar{W}_{a_2} \cdots W_{a_{2k-1}} \right\rangle_{\text{Gauss}} \left\langle \frac{1}{N} \text{tr} W_{a_{2k+1}} \cdots \bar{W}_{a_{2n}} \right\rangle_{\text{Gauss}}.\end{aligned}\quad (3.25)$$

They can be obtained in a usual way by shifting \bar{W}_{2n} and W_1 , respectively.

Equations (3.24) and (3.25) and the cyclic symmetry of the trace result in the equations

$$\lambda G_\lambda(\mathbf{u}) - 1 = \sum_{a=1}^m \mathbf{u}_a \bar{G}_\lambda(\mathbf{u}) \mathbf{u}_a G_\lambda(\mathbf{u}) = \sum_{a=1}^m G_\lambda(\mathbf{u}) \mathbf{u}_a \bar{G}_\lambda(\mathbf{u}) \mathbf{u}_a, \quad (3.26)$$

and

$$\lambda \bar{G}_\lambda(\mathbf{u}) - 1 = \sum_{a=1}^m \sigma(a) \bar{G}_\lambda(\mathbf{u}) \mathbf{u}_a G_\lambda(\mathbf{u}) \mathbf{u}_a = \sum_{a=1}^m \sigma(a) \mathbf{u}_a G_\lambda(\mathbf{u}) \mathbf{u}_a \bar{G}_\lambda(\mathbf{u}). \quad (3.27)$$

These two equations are not independent due to the relation (3.22). One can be obtained from another by the substitution

$$\mathbf{u}_a \rightarrow \bar{\mathbf{u}}_a, \quad (3.28)$$

where \bar{u}_a is given by Eq. (3.23). Their combinatorial interpretation is presented in the Appendix B.

Introducing in addition to Eq. (3.9) the short-hand notations

$$\bar{G}_\lambda \equiv \bar{G}_\lambda(\mathbf{u}), \quad \bar{G}_\lambda^\dagger \equiv \bar{G}_\lambda^\dagger(\mathbf{u}^\dagger), \quad (3.29)$$

one derives from Eqs. (3.26), (3.27) the analog of Eq. (3.11) as

$$\begin{aligned} (\lambda G_\lambda - 1) \mathbf{u}_a^\dagger &= G_\lambda \mathbf{u}_a \bar{G}_\lambda, \\ \mathbf{u}_a (\lambda G_\lambda^\dagger - 1) &= \bar{G}_\lambda^\dagger \mathbf{u}_a^\dagger G_\lambda^\dagger \end{aligned} \quad (3.30)$$

and

$$\begin{aligned} \left(\lambda \bar{G}_\lambda - 1\right) \mathbf{u}_a^\dagger &= \sigma(a) \bar{G}_\lambda \mathbf{u}_a G_\lambda, \\ \mathbf{u}_a \left(\lambda \bar{G}_\lambda^\dagger - 1\right) &= \sigma(a) G_\lambda^\dagger \mathbf{u}_a \bar{G}_\lambda^\dagger. \end{aligned} \quad (3.31)$$

The solution to Eqs. (3.26) and (3.27) can be easily obtained for a pure fermionic model and commutative sources. Summing up Eqs. (3.26) and (3.27), we get

$$G_\lambda = \frac{2}{\lambda} - \bar{G}_\lambda \quad (3.32)$$

and

$$\lambda \bar{G}_\lambda - 1 = \bar{u}^2 \bar{G}_\lambda^2 - \frac{2}{\lambda} \bar{G}_\lambda \quad (3.33)$$

with the solution

$$\bar{G}_\lambda = \frac{1}{\lambda} + \frac{\lambda}{2\vec{u}^2} - \frac{1}{2\lambda\vec{u}^2} \sqrt{\lambda^4 + 4\vec{u}^4} \quad (3.34)$$

and

$$G_\lambda = \frac{1}{\lambda} - \frac{\lambda}{2\vec{u}^2} + \frac{1}{2\lambda\vec{u}^2} \sqrt{\lambda^4 + 4\vec{u}^4}. \quad (3.35)$$

Equations (3.33) and (3.34) for $m = 1$ recover the ones of Ref. [8] while the expansion of Eq. (3.35) in $1/\lambda$ gives Eq. (2.19).

In order to find a formal solution to Eqs. (3.26) and (3.27) in the form of a continued fraction, let us rewrite them as

$$G_\lambda = \frac{1}{\lambda - \mathbf{u}_a \bar{G}_\lambda \mathbf{u}_a} \quad (3.36)$$

and

$$\bar{G}_\lambda = \frac{1}{\lambda - \bar{u}_a G_\lambda \bar{u}_a} \quad (3.37)$$

where $\bar{\mathbf{u}}_a$ is defined by Eq. (3.23). Iterations of this equations lead to the following analog of (3.14)

$$G_\lambda(\mathbf{u}) = \frac{1}{\lambda - \mathbf{u}_{a_1} \frac{1}{\lambda - \bar{\mathbf{u}}_{a_2} \frac{1}{\lambda - \mathbf{u}_{a_3} \frac{1}{\lambda - \bar{\mathbf{u}}_{a_4} \frac{1}{\vdots}}}} \mathbf{u}_{a_1}} \quad (3.38)$$

and

$$\bar{G}_\lambda(\mathbf{u}) = \frac{1}{\lambda - \bar{\mathbf{u}}_{a_1} \frac{1}{\lambda - \mathbf{u}_{a_2} \frac{1}{\lambda - \bar{\mathbf{u}}_{a_3} \frac{1}{\lambda - \mathbf{u}_{a_4} \frac{1}{\vdots}}}} \bar{\mathbf{u}}_{a_1}}. \quad (3.39)$$

Here \mathbf{u} and $\bar{\mathbf{u}}$ interchange in the consequent lines of the continued fractions.

The quantities on the LHS's of Eqs. (3.30) and (3.31) read explicitly

$$\mathbf{u}_a \left(\lambda G_\lambda^\dagger - 1 \right) = \sum_{n=1}^{\infty} \frac{1}{\lambda^{2n}} \sum_{a_2, \dots, a_{2n}=1}^m \mathbf{u}_{a_2}^\dagger \cdots \mathbf{u}_{a_{2n-1}}^\dagger \mathbf{u}_{a_{2n}}^\dagger \left\langle \frac{1}{N} \text{tr} W_a \bar{W}_{a_2} \cdots W_{a_{2n-1}} \bar{W}_{a_{2n}} \right\rangle_{\text{Gauss}} \quad (3.40)$$

and

$$\left(\lambda \bar{G}_\lambda - 1\right) \mathbf{u}_a^\dagger = \sum_{n=1}^{\infty} \frac{1}{\lambda^{2n}} \sum_{a_1, a_2, \dots, a_{2n-1}=1}^m \mathbf{u}_{a_1} \mathbf{u}_{a_2} \cdots \mathbf{u}_{a_{2n-1}} \left\langle \frac{1}{N} \text{tr } \bar{W}_{a_1} W_{a_2} \cdots \bar{W}_{a_{2n-1}} W_a \right\rangle_{\text{Gauss}}. \quad (3.41)$$

The generating function (2.26) is determined by

$$\langle \Omega | G_\lambda(\mathbf{u}) \bar{G}_\lambda(\mathbf{u}^\dagger) | \Omega \rangle = \langle \Omega | \bar{G}_\lambda(\mathbf{u}) G_\lambda(\mathbf{u}^\dagger) | \Omega \rangle = c + c(m_b - m_f)M(c) \quad (3.42)$$

where m_b and m_f are the numbers of bosonic and fermionic components of W_a , respectively, and $\lambda^2 = 1/c$. This formula follows from the fact that the LHS reproduces the contraction of indices as in Eq. (2.25) using Eqs. (3.1) and (3.2).

3.3 Supersymmetric case

For the supersymmetric case $m_b = m_f = 1$ and Eq. (3.42) does not determine the meander numbers. One should use, instead, Eq. (2.37) or Eq. (2.38), where there is no summation over one of the indices, to get the principle meander numbers.

Let us denote the components of \mathbf{u}_a as

$$\mathbf{u}_a = (u, v). \quad (3.43)$$

Equations (3.40) and (3.41) then result in

$$M(\frac{1}{\lambda^2}) = \lambda^2 \langle \Omega | \bar{G}_\lambda u^\dagger u G_\lambda^\dagger | \Omega \rangle = -\lambda^2 \langle \Omega | \bar{G}_\lambda v^\dagger v G_\lambda^\dagger | \Omega \rangle \quad (3.44)$$

or alternatively

$$M(\frac{1}{\lambda^2}) = \lambda^2 \langle \Omega | G_\lambda u^\dagger u \bar{G}_\lambda^\dagger | \Omega \rangle = -\lambda^2 \langle \Omega | G_\lambda v^\dagger v \bar{G}_\lambda^\dagger | \Omega \rangle. \quad (3.45)$$

The equality sign between the two expressions on the RHS of Eq. (3.44) or Eq. (3.45) is due to the supersymmetry. This can be shown using the completeness condition (3.3) which takes in the supersymmetric case the form

$$u^\dagger u + v^\dagger v = 1 - |\Omega\rangle \langle \Omega| . \quad (3.46)$$

Inserting this, say, between G_λ and \bar{G}_λ^\dagger , we get

$$\lambda^2 G_\lambda u^\dagger u \bar{G}_\lambda^\dagger = -\lambda^2 G_\lambda v^\dagger v \bar{G}_\lambda^\dagger + \lambda^2 G_\lambda \bar{G}_\lambda^\dagger - |\Omega\rangle \langle \Omega| . \quad (3.47)$$

Taking the vacuum expectation value of this formula and remembering that

$$\lambda^2 \langle \Omega | G_\lambda \bar{G}_\lambda^\dagger | \Omega \rangle = 1 \quad (3.48)$$

due to the supersymmetry, we prove the statement.

Using Eqs. (3.30) and (3.31), Eqs. (3.45) and (3.44) can be rewritten as

$$M(\frac{1}{\lambda^2}) = \langle \Omega | G_\lambda u \bar{G}_\lambda G_\lambda^\dagger u^\dagger \bar{G}_\lambda^\dagger | \Omega \rangle = \langle \Omega | G_\lambda v \bar{G}_\lambda G_\lambda^\dagger v^\dagger \bar{G}_\lambda^\dagger | \Omega \rangle \quad (3.49)$$

and

$$M(\frac{1}{\lambda^2}) = \langle \Omega | \bar{G}_\lambda u G_\lambda \bar{G}_\lambda^\dagger u^\dagger G_\lambda^\dagger | \Omega \rangle = \langle \Omega | \bar{G}_\lambda v G_\lambda \bar{G}_\lambda^\dagger v^\dagger G_\lambda^\dagger | \Omega \rangle . \quad (3.50)$$

This representation of the generating function for the principle meander numbers is convenient for an iterative procedure.

To perform calculations order by order in $c = 1/\lambda^2$, it looks reasonable to try to use the fact that there are only two non-commutative variables and to expand in v . This is a standard trick of dealing with two non-commutative variables in the Hermitean two-matrix model [20, 21].

We introduce, therefore, the expansions

$$\begin{aligned} G_\lambda &= \sum_{n=0}^{\infty} G_\lambda^{(n)} , \\ \bar{G}_\lambda &= \sum_{n=0}^{\infty} (-1)^n G_\lambda^{(n)} , \end{aligned} \quad (3.51)$$

where $(-1)^n$ in the expansion of \bar{G}_λ is due to Eqs. (3.22) (3.23). We have

$$G_\lambda^{(0)} = \frac{\lambda - \sqrt{\lambda^2 - 4u^2}}{2u^2} \quad (3.52)$$

while $G_\lambda^{(n)}$ involves v exactly $2n$ times. As we shall see below,

$$\begin{aligned} G_\lambda^{(n)} &\sim \frac{1}{\lambda^{2n+1}} \quad \text{for } n \text{ odd} , \\ G_\lambda^{(n)} &\sim \frac{1}{\lambda^{2n+3}} \quad \text{for } n \geq 2 \text{ even} . \end{aligned} \quad (3.53)$$

Therefore, a contribution to M_n arises in Eq. (3.45) at most from $G_\lambda^{(n)}$ for odd n and from $G_\lambda^{(n-1)}$ for even n .

The functions $G_\lambda^{(n)}$ can be found recursively from Eq. (3.36) which takes the form

$$\sum_{n=0}^{\infty} G_\lambda^{(n)} = G_\lambda^{(0)} \frac{1}{1 - \sum_{n=1}^{\infty} (-1)^n u G_\lambda^{(n)} u G_\lambda^{(0)} - \sum_{n=0}^{\infty} (-1)^n v G_\lambda^{(n)} v G_\lambda^{(0)}}. \quad (3.54)$$

At each step one has to solve the equation

$$G_\lambda^{(n)} = (-1)^n G_\lambda^{(0)} u G_\lambda^{(n)} u G_\lambda^{(0)} + A_n \quad (3.55)$$

with some A_n whose solution is given by

$$\begin{aligned} G_\lambda^{(2p-1)} &= \sum_{l=0}^{\infty} (-1)^l \left(G_\lambda^{(0)} u \right)^l A_{2p-1} \left(G_\lambda^{(0)} u \right)^l, \\ G_\lambda^{(2p)} &= \sum_{l=0}^{\infty} \left(G_\lambda^{(0)} u \right)^l A_{2p} \left(G_\lambda^{(0)} u \right)^l. \end{aligned} \quad (3.56)$$

where $p \geq 1$. Both formulas can be rewritten in a unique way via the contour integral

$$G_\lambda^{(n)} = \oint \frac{d\omega}{2\pi i} \frac{1}{\left(\omega - (-1)^n G_\lambda^{(0)} u \right)} A_n \frac{1}{\left(1 - \omega G_\lambda^{(0)} u \right)}. \quad (3.57)$$

We shall also introduce the obvious short-hand notation for RHS's of Eqs. (3.56) or (3.57) as the brackets

$$G_\lambda^{(n)} = \{ A_n \}. \quad (3.58)$$

The functions A_n can be found recursively from Eq. (3.54). Few lower ones read explicitly

$$\begin{aligned} A_1 &= G_\lambda^{(0)} v G_\lambda^{(0)} v G_\lambda^{(0)}, \\ A_2 &= -G_\lambda^{(0)} v G_\lambda^{(1)} v G_\lambda^{(0)} + G_\lambda^{(1)} \frac{1}{G_\lambda^{(0)}} G_\lambda^{(1)}, \\ A_3 &= G_\lambda^{(0)} v G_\lambda^{(2)} v G_\lambda^{(0)} + G_\lambda^{(2)} \frac{1}{G_\lambda^{(0)}} G_\lambda^{(1)} + G_\lambda^{(1)} \frac{1}{G_\lambda^{(0)}} G_\lambda^{(2)} - G_\lambda^{(1)} \frac{1}{G_\lambda^{(0)}} G_\lambda^{(1)} \frac{1}{G_\lambda^{(0)}} G_\lambda^{(1)}. \end{aligned} \quad (3.59)$$

They obviously satisfy the equation

$$\lambda A_n v^\dagger = \sum_{k=0}^{n-1} (-1)^k G_\lambda^{(n-1-k)} v G_\lambda^{(k)} \quad (3.60)$$

and analogously

$$\lambda v A_n^\dagger = \sum_{k=0}^{n-1} (-1)^k G_\lambda^{\dagger(k)} v^\dagger G_\lambda^{\dagger(n-1-k)}, \quad (3.61)$$

which can be obtained substituting the expansion (3.51) into Eq. (3.30) (or Eq. (3.31)) and using the fact that

$$G_\lambda^{(n)} v^\dagger = A_n v^\dagger \quad (3.62)$$

due to Eq. (3.56).

Using Eq. (3.58), we get from Eq. (3.59) for $G_\lambda^{(n)}$'s

$$\begin{aligned}
G_\lambda^{(1)} &= G_\lambda^{(0)} \{v G_\lambda^{(0)} v\} G_\lambda^{(0)}, \\
G_\lambda^{(2)} &= -G_\lambda^{(0)} \{v G_\lambda^{(0)} \{v G_\lambda^{(0)} v\} G_\lambda^{(0)} v\} G_\lambda^{(0)} + G_\lambda^{(0)} \{ \{v G_\lambda^{(0)} v\} G_\lambda^{(0)} \{v G_\lambda^{(0)} v\} \} G_\lambda^{(0)}, \\
G_\lambda^{(3)} &= -G_\lambda^{(0)} \{v G_\lambda^{(0)} \{v G_\lambda^{(0)} \{v G_\lambda^{(0)} v\} G_\lambda^{(0)} v\} G_\lambda^{(0)} v\} G_\lambda^{(0)} \\
&\quad + G_\lambda^{(0)} \{v G_\lambda^{(0)} \{v G_\lambda^{(0)} v\} G_\lambda^{(0)} \{v G_\lambda^{(0)} v\} G_\lambda^{(0)} v\} G_\lambda^{(0)} \\
&\quad - G_\lambda^{(0)} \{ \{v G_\lambda^{(0)} v\} G_\lambda^{(0)} \{v G_\lambda^{(0)} \{v G_\lambda^{(0)} v\} G_\lambda^{(0)} v\} \} G_\lambda^{(0)} \\
&\quad + G_\lambda^{(0)} \{ \{v G_\lambda^{(0)} v\} G_\lambda^{(0)} \{v G_\lambda^{(0)} v\} G_\lambda^{(0)} \{v G_\lambda^{(0)} v\} \} G_\lambda^{(0)} \\
&\quad - G_\lambda^{(0)} \{ \{v G_\lambda^{(0)} \{v G_\lambda^{(0)} v\} G_\lambda^{(0)} v\} G_\lambda^{(0)} \{v G_\lambda^{(0)} v\} \} G_\lambda^{(0)}, \tag{3.63}
\end{aligned}$$

where we have used the fact that $G_\lambda^{(0)}$ commutes with a bracket but not with v . There are no problems here with the order of brackets which act like in TeX: they immediately contract each other when it is possible. This contraction is associated with the summation (3.56) (or the integration (3.57)). The general $G_\lambda^{(n)}$ contains C_n (the Catalan number given by Eq. (2.5)) terms of this kind with alternating signs. Some rules for representing a general term can be formulated which resemble the Wick pairing of bilinear combinations of v 's.

It is worth noting that most of the above formulas would look quite similar for the case of just 2 bosonic fields, *i.e.* of $m = 2$, while there will be no alternative signs for 2 bosons. Due to these minus signs, the supersymmetric case is somewhat simpler than the $m = 2$ one. Say, the leading-order terms v^4 are cancelled in Eq. (3.59) for A_2 and in Eq. (3.63) for $G_\lambda^{(2)}$. This is a reflection of the general property (2.19) of Gaussian averages of the fermionic matrices. It is the reason why the leading order term vanishes in Eq. (3.53) for even n . In addition some relations are imposed by the supersymmetry. The simplest one follows from the substitution of the expansion (3.51) into Eq. (3.48) which yields

$$\lambda^2 \sum_{n=0}^{\infty} (-1)^n \langle \Omega | G_\lambda^{(n)} G_\lambda^{\dagger(n)} | \Omega \rangle = 1. \tag{3.64}$$

Only the diagonal terms survive here due to the definitions (3.1), (3.2).

In order to calculate M_n , we substitute the expansion (3.51) into Eq. (3.45) which gives

$$M\left(\frac{1}{\lambda^2}\right) = \lambda^2 \sum_{n=1}^{\infty} (-1)^{(n-1)} \langle \Omega | G_\lambda^{(n)} v^\dagger v G_\lambda^{\dagger(n)} | \Omega \rangle. \tag{3.65}$$

We can also rewrite the RHS as

$$M\left(\frac{1}{\lambda^2}\right) = \lambda^2 \sum_{n=1}^{\infty} (-1)^{(n-1)} \langle \Omega | A_n v^\dagger v A_n^\dagger | \Omega \rangle \tag{3.66}$$

due to Eq. (3.62).

It is easy to calculate the term with $n = 1$ in Eq. (3.65) (or Eq. (3.66)). Using Eq. (3.63), we get

$$G_\lambda^{(1)} v^\dagger = \frac{1}{\lambda} G_\lambda^{(0)} v G_\lambda^{(0)}, \quad v G_\lambda^{\dagger(1)} = \frac{1}{\lambda} G_\lambda^{\dagger(0)} v^\dagger G_\lambda^{\dagger(0)}, \quad (3.67)$$

so that Eqs. (3.1), (3.2) and (3.52) result in

$$M\left(\frac{1}{\lambda^2}\right) = c^2 \left(\sum_{k=0}^{\infty} c^{2k} C_k^2 \right)^2 + \lambda^2 \sum_{n=2}^{\infty} (-1)^{(n-1)} \langle \Omega | G_\lambda^{(n)} v^\dagger v G_\lambda^{\dagger(n)} | \Omega \rangle. \quad (3.68)$$

The first term on the RHS recovers $M_1 = 1$ and $M_2 = 2$ while the contribution of the next terms are controlled by Eq. (3.53).

Let us demonstrate of how this iterative procedure works by the explicit calculation up to $n = 3$ when at most $G_\lambda^{(3)}$ is essential. We get from Eq. (3.63)

$$\begin{aligned} G_\lambda^{(2)} v^\dagger &= \frac{-vuv^2u + uv^2uv}{\lambda^7} + \mathcal{O}\left(\frac{1}{\lambda^9}\right), & v G_\lambda^{\dagger(2)} &= \frac{v^\dagger u^\dagger v^{\dagger 2} u^\dagger - u^\dagger v^{\dagger 2} u^\dagger v^\dagger}{\lambda^7} + \mathcal{O}\left(\frac{1}{\lambda^9}\right), \\ G_\lambda^{(3)} v^\dagger &= -\frac{v^6}{\lambda^7} + \mathcal{O}\left(\frac{1}{\lambda^9}\right), & v G_\lambda^{\dagger(3)} &= -\frac{v^{\dagger 6}}{\lambda^7} + \mathcal{O}\left(\frac{1}{\lambda^9}\right). \end{aligned} \quad (3.69)$$

The substitution into Eq. (3.68) recovers $M_3 = 8$.

A most difficult part of the described general iterative procedure is to calculate $\langle \Omega | G_\lambda^{(n)} v^\dagger v G_\lambda^{\dagger(n)} | \Omega \rangle$ which involves C_n^2 terms for $n > 1$. There is no problem to calculate a contribution from each individual term by using the formula

$$\begin{aligned} &\langle \Omega | f_0(u) v f_1(u) v \cdots v f_n(u) g_n(u^\dagger) v^\dagger \cdots v^\dagger g_1(u^\dagger) v^\dagger g_0(u^\dagger) | \Omega \rangle \\ &= \langle \Omega | f_0(u) g_0(u^\dagger) | \Omega \rangle \langle \Omega | f_1(u) g_1(u^\dagger) | \Omega \rangle \cdots \langle \Omega | f_n(u) g_n(u^\dagger) | \Omega \rangle, \end{aligned} \quad (3.70)$$

which is a consequence of Eqs. (3.1) and (3.2). When calculating order by order in $c = 1/\lambda^2$, it is more convenient first to collect similar terms in $G_\lambda^{(n)} v^\dagger$ and $v G_\lambda^{\dagger(n)}$ to the given order c^n . After this each term in $G_\lambda^{(n)} v^\dagger$ is orthogonal to all terms in $v G_\lambda^{\dagger(n)}$ except for one. Whether or not this kind of an iterative procedure could be useful for deriving recurrence relation between meander numbers would depend on a possibility to reexpand (3.59) in such orthogonal terms in the general case.

We see now the difference between the expansion in v for the meander problem, described in this Subsection, and that for the two-matrix model with a polynomial potential [20]. The latter reduces to an algebraic equation while the former does not. This differs the matrix models describing the meander numbers from the Kazakov–Migdal model whose solution can be described in the language of the two-matrix model [22].

4 Discussion

We have considered in this paper the meander problem which results in more complicated matrix models than those solved before. It belongs to the same generic class of problems of words as, say, the large- N QCD in $D = 4$ but is presumably simpler.

We have introduced a new representation of the meander numbers via the general complex matrix model. This allowed us to consider the supersymmetric matrix model which includes both bosonic and fermionic matrices and describes the principal meanders. This model looks simpler than a pure bosonic model and one can try to solve it by using supersymmetry Ward identities.

Using non-commutative sources, we have reformulated the meander problem as that of averaging in a Boltzmannian Fock space whose annihilation and creation operators obey the Cuntz algebra and have shown the equivalence with the combinatorial approach based on the arch statistics. The averaging expression is represented in the form of a product of two continued fractions but further progress requires their better understanding as functions of the non-commutative variables. One of the possible approaches can be based on the standard procedure of disentangling via the path integral.

We have discussed also the relation between the matrix models describing the meander problem and the Kazakov–Migdal model on a D -dimensional lattice. The words are the same both for the meander problem and for the Kazakov–Migdal model while the only difference resides in the meaning of nonvanishing words. This relation could give a hint on how to solve the meander problem.

We have demonstrated how the solution of the Kazakov–Migdal model with the quadratic potential can be obtained using the theorem of addition of free random variables. We have shown that this approach does not work for the meander problem even within the interaction representation since the variables are not free for this case so that the theorem of addition is not applicable. This observation casts doubts that a solution of large- N QCD can be obtained in the language of a masterfield given by free random variables.

The supersymmetric matrix models of the type discussed in this paper for the meander problem are novel ones. It is worth studying them in connection with other physical applications, in particular, with a discretization of super-Riemann surfaces and superstrings.

Acknowledgments

We are grateful to J. Ambjørn, L. Chekhov, P. Cvitanović, C. Kristjansen, G. Semenoff, K. Zarembo, and J.-B. Zuber for useful discussions. The research described in this publication was supported in part by the International Science Foundation under Grant MF-7300.

Appendix A Derivation of Eq. (2.39) via free random variables

Equation (2.39) can be derived using free random variables as follows. First of all let us note that

$$\begin{aligned} \#_n &= \# \text{ of nonvanishing terms in } \left\langle \frac{1}{N} \text{tr} \left(\sum_{a,b=1}^m W_a W_b^\dagger \right)^n \right\rangle_{\text{Gauss}} \\ &= \# \text{ of nonvanishing terms in } \left\langle \frac{1}{N} \text{tr} \left(\sum_{i=1}^m A_i \right)^{2n} \right\rangle_{\text{Gauss}} \end{aligned} \quad (\text{A.1})$$

where the matrices A_i are Hermitean. The simplest nontrivial example is that of $m = 2$, when the LHS has for $n = 2$ six nonvanishing terms of the type of $W_1 W_1^\dagger W_1 W_1^\dagger$, $W_1 W_2^\dagger W_2 W_1^\dagger$, $W_1 W_1^\dagger W_2 W_2^\dagger$, $W_2 W_2^\dagger W_2 W_2^\dagger$, $W_2 W_1^\dagger W_1 W_2^\dagger$, $W_2 W_2^\dagger W_1 W_1^\dagger$. The RHS has six nonvanishing terms of the type of A^4 , $A^2 B^2$, $AB^2 A$, B^4 , $B^2 A^2$, $BA^2 B$, where we have used the notations $A_1 \equiv A$, $A_2 \equiv B$ — the same as in the Table 3.

If we now introduce free random variables \hat{A}_i , such that

$$\begin{aligned} \langle \Omega | \hat{A}_i^{2n} | \Omega \rangle &= 1, \\ \langle \Omega | \hat{A}_i^{2n+1} | \Omega \rangle &= 0, \end{aligned} \quad (\text{A.2})$$

then

$$\left\langle \Omega \left| \left(\sum_{i=1}^m \hat{A}_i \right)^{2n} \right| \Omega \right\rangle = \#_n \quad (\text{A.3})$$

since the vacuum average of each nonvanishing term equals one. The resolvent for each \hat{A}_i reads

$$R_i(z) \equiv \sum_{n=0}^{\infty} \frac{\langle \Omega | \hat{A}_i^n | \Omega \rangle}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{z^{2n+1}} = \frac{z}{z^2 - 1}. \quad (\text{A.4})$$

The inverse function to (A.4) is given by

$$A_i(z) \equiv R_i^{-1}(z) = \frac{\sqrt{1 + 4z^2} + 1}{2z}. \quad (\text{A.5})$$

Then the master-field operator can explicitly be constructed as [18, 19]

$$\hat{A}_i(a, a^\dagger) = a + \frac{\sqrt{1 + 4(a^\dagger)^2} - 1}{2a^\dagger} = a + a^\dagger - (a^\dagger)^3 + \dots \quad (\text{A.6})$$

where the RHS is understood as the expansion around zero.

The theorem of addition of free random variables states that if \hat{A}_i 's are free random variables and so they are representable as

$$\hat{A}_i = a_i + f_i(a_i^\dagger), \quad (\text{A.7})$$

where f_i 's are functions analytic around zero, then

$$\left\langle \Omega \left| \left(\sum_i \hat{A}_i \right)^n \right| \Omega \right\rangle = \left\langle \Omega \left| \left(a + \sum_i f_i(a^\dagger) \right)^n \right| \Omega \right\rangle. \quad (\text{A.8})$$

Using this theorem, we get for the m matrices

$$A(z) = \frac{1}{z} + m \left(A_i(z) - \frac{1}{z} \right) = \frac{m\sqrt{1+4z^2} + 2 - m}{2z}. \quad (\text{A.9})$$

The resolvent is given by the inverse function

$$R(z) = A^{-1}(z) = \frac{(2-m)z + m\sqrt{z^2 - 4(m-1)}}{2(z^2 - m^2)}. \quad (\text{A.10})$$

Equation (2.39) can be reproduced substituting $z = 1/c$ and dividing by c .

As far as the Kazakov–Migdal model is concerned, Eq. (2.46) results in the following representation of the correlator (2.42) via the words built up of the $m = 2D$ unitary matrices:

$$\left\langle \frac{1}{N} \text{tr} \phi^2(0) \right\rangle = \sum_{n=0}^{\infty} c^{2n} \sum_{a_1, a_2, \dots, a_{2n}=1}^{2D} \left\langle \frac{1}{N} \text{tr} (U_{a_1} U_{a_2}^\dagger \dots U_{a_{2n-1}} U_{a_{2n}}^\dagger) \right\rangle_{\text{Haar measure}}^2. \quad (\text{A.11})$$

Since the meaning of each word is either one or zero, the square on the RHS is not essential. Equation (2.39) with $m = 2D$ then yields

$$\left\langle \frac{1}{N} \text{tr} \phi^2(0) \right\rangle = \frac{D\sqrt{1-4(2D-1)c^2} - D + 1}{1-4D^2c^2} \quad (\text{A.12})$$

which reproduces the solution of Ref. [10].

We discuss in the Appendix C that the approach, which is described here for the Kazakov–Migdal model, does not work for the meander problem since the random variables are not free in the latter case.

Appendix B Combinatorial interpretation of the results of Sect. 3

The representation of Sect. 3 of the meanders via non-commutative random variables can be alternatively derived pure combinatorially which clarifies the relation between our approach and that of Ref. [6].

Equation (3.6) can be derived combinatorially as follows. Let us denote by \mathcal{G}_n the sum of all possible arch configurations of order n with m colorings. For example, \mathcal{G}_1 for $m = 2$ is given by

$$\mathcal{G}_1 = \text{arch} + \text{arch} , \quad (\text{B.1})$$

where the two terms on the RHS differ by the coloring. Picking up the leftmost arch, we have the following recurrence relation for \mathcal{G}_n 's:

$$\mathcal{G}_n = \sum_{k=0}^{n-1} \sum_{\substack{\text{colors} \\ \text{of } \cap}} \text{arch}(\mathcal{G}_k) \mathcal{G}_{n-k-1} , \quad \mathcal{G}_0 = \emptyset . \quad (\text{B.2})$$

Let us analogously denote by \mathcal{G}_n^\dagger the sum of all possible upside-down arch configurations of order n with m colorings. For example, \mathcal{G}_1^\dagger for $m = 2$ is given by

$$\mathcal{G}_1^\dagger = \text{upside-down arch} + \text{upside-down arch} . \quad (\text{B.3})$$

\mathcal{G}_n^\dagger obeys the recurrence relation which is similar to Eq. (B.2).

Let us now introduce the “multiplication” \circ of arch configurations. Its meaning is evident from the following examples:

$$\begin{aligned} \text{arch} \circ \text{arch} &= \text{two arches joined at a point} , \\ \text{arch} \circ \text{upside-down arch} &= \emptyset . \end{aligned}$$

We have then the relation

$$\mathcal{G}_n \circ \mathcal{G}_n^\dagger = \left\{ \begin{array}{l} \text{the sum of all possible} \\ \text{multi-component meanders} \end{array} \right\} , \quad (\text{B.4})$$

and

$$\# \text{ of terms in } \mathcal{G}_n \circ \mathcal{G}_n^\dagger = \sum_{k=1}^n M_n^{(k)} m^k , \quad (\text{B.5})$$

since each loop in the meander can be colored with m colors.

The “multiplication” \circ can naturally be represented in the operator language as follows. Let us introduce m non-commuting variables \mathbf{u}_a ($a = 1, \dots, m$), one for each color in the arch configurations. Let us associate with each arch configuration a word made up of \mathbf{u}_a 's in a way, which is evident from the following example:

$$\text{arch} \longleftrightarrow \mathbf{u}_1 \mathbf{u}_1 \mathbf{u}_2 \mathbf{u}_2 , \quad \text{if} \quad \mathbf{u}_1 \leftrightarrow \text{—} \quad \text{and} \quad \mathbf{u}_2 \leftrightarrow \text{—} .$$

Analogously, we associate words, made up of \mathbf{u}_a^\dagger 's, with the upside-down arch configurations as in the example

$$\cup \cup \longleftrightarrow (\mathbf{u}_1 \mathbf{u}_1 \mathbf{u}_2 \mathbf{u}_2)^\dagger = \mathbf{u}_2^\dagger \mathbf{u}_2^\dagger \mathbf{u}_1^\dagger \mathbf{u}_1^\dagger, \quad \text{if} \quad \mathbf{u}_1 \leftrightarrow \text{---} \text{ and } \mathbf{u}_2 \leftrightarrow \text{---} .$$

We have from Eq. (B.2) the following equation in terms of \mathbf{u}_a 's:

$$G_n(\mathbf{u}) = \sum_{k=0}^{n-1} \sum_{a=1}^m \mathbf{u}_a G_k(\mathbf{u}) \mathbf{u}_a G_{n-k-1}(\mathbf{u}), \quad (\text{B.6})$$

where $G_n(\mathbf{u})$ stands for the sum of the words of order n . Introducing the generating function (cf. (3.4))

$$G_\lambda(\mathbf{u}) = \sum_{n=0}^{\infty} G_n(\mathbf{u}) \lambda^{-2n-1}, \quad (\text{B.7})$$

Eq. (B.6) can be rewritten in the form (3.6).

There is the correspondence

$$\mathcal{A} \circ \mathcal{B} \longleftrightarrow \langle \Omega | A(\mathbf{u}) B(\mathbf{u}^\dagger) | \Omega \rangle. \quad (\text{B.8})$$

Using this correspondence, Eqs. (B.5) and (B.7), we arrive at Eq. (3.5).

Equation (3.26) can be interpreted in the same way. As an example, let us consider the supersymmetric case of $m = 2$, when $\mathbf{u}_a = (u, v)$. The analog of Eq. (B.2) for this case reads

$$\begin{aligned} \mathcal{G}_0 &= \emptyset, \\ \mathcal{G}_n &= \sum_{k=0}^{n-1} \left[\overbrace{\mathcal{G}_k}^{\text{light}} \mathcal{G}_{n-k-1} + \overbrace{\mathcal{G}_k}^{\text{dark}} \mathcal{G}_{n-k-1} \right], \end{aligned} \quad (\text{B.9})$$

where

$$\begin{aligned} \mathcal{G}_n &= \mathcal{G}_n(\cap, \cap), \\ \bar{\mathcal{G}}_n &= \mathcal{G}_n(\cap, -\cap). \end{aligned} \quad (\text{B.10})$$

Let us introduce the projector \mathcal{P} , such that it picks up the arch configurations whose rightmost arch is of light color. For example,

$$\mathcal{P} \left[\cap + \overbrace{\cap}^{\text{light}} \right] = \cap.$$

Then, we have

$$\left\{ \begin{array}{l} \text{the sum of all principle} \\ \text{meanders of order } n \end{array} \right\} = \mathcal{P} [\mathcal{G}_n] \circ \mathcal{P} [\bar{\mathcal{G}}_n^\dagger]. \quad (\text{B.11})$$

The projector \mathcal{P} corresponds in the operator language to the insertion of $u^\dagger u$ in the Eq. (3.44).

Appendix C Comment on freeness of random variables for the meander problem

We demonstrated in the Appendix A how the Kazakov–Migdal model can be solved via free random variables. We comment here that this approach does not work for the meander problem since the random variables are not free for this case.

The meander numbers can be represented by the vacuum average of certain non-commuting variables as

$$\sum_{k=1}^n M_n^{(k)} m^k = \frac{1}{N^2} \left\langle \text{Tr} \left(\sum_{i=1}^m A_i \otimes A_i \right)^n \right\rangle_{\text{Gauss}} = \left\langle \Omega \left| \left(\sum_{i=1}^m \hat{A}_i \hat{\tilde{A}}_i \right)^n \right| \Omega \right\rangle, \quad (\text{C.1})$$

where the operators \hat{A}_i 's and $\hat{\tilde{A}}_i$'s under the vacuum averaging are

$$\hat{A}_i = a_i + a_i^\dagger, \quad \hat{\tilde{A}}_i = \tilde{a}_i + \tilde{a}_i^\dagger. \quad (\text{C.2})$$

Here the two sets of operators a_i 's and \tilde{a}_i 's obey the Cuntz's algebra

$$a_i a_j^\dagger = \delta_{ij}, \quad \tilde{a}_i \tilde{a}_j^\dagger = \delta_{ij} \quad (\text{C.3})$$

independently of each other while all the operators with tildes commute with all the operators without tildes. The equivalence of the two representations in Eq. (C.1) is evident from the factorization

$$\langle \text{Tr}(A \otimes B) \rangle \equiv \langle \text{tr } A \text{ tr } B \rangle = \langle \text{tr } A \rangle \langle \text{tr } B \rangle \quad (\text{C.4})$$

at large N .

Equation (C.1) involves the direct product of matrices which is represented as the product of two independent operators \hat{A}_i and $\hat{\tilde{A}}_i$. The non-commuting variables of the type $\hat{A}_i \hat{\tilde{A}}_i$ are not free random variables of Ref. [17] since they do not satisfy the defining axiom of free random variables. This can be seen by considering the average

$$\begin{aligned} & \left\langle \Omega \left| \left[(\hat{A}\hat{\tilde{A}})^2 - \langle (\hat{A}\hat{\tilde{A}})^2 \rangle \right] \left[(\hat{B}\hat{\tilde{B}})^2 - \langle (\hat{B}\hat{\tilde{B}})^2 \rangle \right] \right. \right. \\ & \quad \left. \times \left[(\hat{A}\hat{\tilde{A}})^2 - \langle (\hat{A}\hat{\tilde{A}})^2 \rangle \right] \left[(\hat{B}\hat{\tilde{B}})^2 - \langle (\hat{B}\hat{\tilde{B}})^2 \rangle \right] \right| \Omega \right\rangle \stackrel{?}{=} 0 \end{aligned} \quad (\text{C.5})$$

which would vanish if they were free random variables.

It is easy to see by direct calculation that the expression (C.5) does not vanish. We rewrite it as

$$\begin{aligned} & \left\langle \Omega \left| \left[(\hat{A}\hat{\tilde{A}})^2 - 1 \right] \left[(\hat{B}\hat{\tilde{B}})^2 - 1 \right] \left[(\hat{A}\hat{\tilde{A}})^2 - 1 \right] \left[(\hat{B}\hat{\tilde{B}})^2 - 1 \right] \right| \Omega \right\rangle \\ & = \left\langle \Omega \left| \hat{A}^2 \hat{B}^2 \hat{A}^2 \hat{B}^2 \right| \Omega \right\rangle^2 - \left\langle \Omega \left| \hat{A}^4 \right| \Omega \right\rangle^2 - \left\langle \Omega \left| \hat{B}^4 \right| \Omega \right\rangle^2 + 1 \\ & = 3^2 - 2^2 - 2^2 + 1 = 2 \end{aligned} \quad (\text{C.6})$$

since

$$\left\langle \Omega \left| (\hat{A}\hat{\tilde{A}})^2 \right| \Omega \right\rangle = \left\langle \Omega \left| (\hat{B}\hat{\tilde{B}})^2 \right| \Omega \right\rangle = 1. \quad (\text{C.7})$$

The squares on the RHS of Eq. (C.6) are due to analogous contributions from the operators with tildes.

The nonvanishing RHS of Eq. (C.6) means that the variables $\hat{A}_i\hat{\tilde{A}}_i$ are *not* free. This is, as is already mentioned, a consequence of the direct product of matrices which results in the product of \hat{A}_i and $\hat{\tilde{A}}_i$ and leads, in turn, to the squares in Eq. (C.6). If there are no squares, the axiom of freeness is obviously satisfied. The same is true for the supersymmetric case as well.

The example of this Appendix shows that random variables are not necessarily free in (multi-) matrix models with complicated interaction even within the framework of the interaction representation. The simplest average for the meander problem which violates the axiom of freeness is that (C.5). For this reason, the theorem of addition with

$$R_i(z) \equiv \sum_{n=0}^{\infty} \frac{\left\langle \Omega \left| (\hat{A}_i\hat{\tilde{A}}_i)^n \right| \Omega \right\rangle}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{C_n^2}{z^{2n+1}}, \quad (\text{C.8})$$

where C_n are given by Eq. (2.5), can be used, quite similarly to the Appendix A, to describe the meander numbers only up to $n = 3$ but fails to reproduce M_n for $n \geq 4$.

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